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Journal of Sound and Vibration 273 (2004) 837–855
JOURNAL OF SOUND AND VIBRATION
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System response to partially known input

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Received 16 December 2002; accepted 4 May 2003

Abstract

The available information on the random parameters in the definition of a variety of engineering, physics, biological, and other stochastic problems is usually insufficient for specifying the probability law of these parameters. Generally, there is a collection of probabilistic models consistent with the available information, referred to as the class of competing models. The paper examines the output of perfectly known deterministic systems subjected to partially specified input processes. It is shown that the class $\mathscr{C}_{in}$ of competing models for input can have many members and that the output properties may depend strongly on the particular member in $\mathscr{C}_{in}$ used to represent the input. Since the available information on the input is usually incomplete, it is rarely possible to find uniquely the output properties relevant for a stochastic problem. Simple linear and non-linear systems with quasi-static and dynamic output are used to illustrate the flow of information from input to output and assess the sensitivity of some output properties to the input model. The examples in the paper also show that estimates of a system performance can be inaccurate if based on input models selected by heuristic considerations.
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1. Introduction

Consider a linear or non-linear deterministic system with input $\mathbf{X}$ and output $\mathbf{Y}$, where $\mathbf{X}(t)$ and $\mathbf{Y}(t)$, $t \in [t_0, t_1]$, are $\mathbb{R}^d$- and $\mathbb{R}^{d'}$-valued processes, respectively. Suppose that the system is perfectly known and that the available information on the input is limited to data and physics. Physics may include some prior information inferred from the analysis of similar stochastic problems. The objective is to find some properties $\mathscr{P}^*$ for $\mathbf{Y}$ that may or may not characterize the probability law of this process completely. Let $\mathscr{H}^*$ denote the class of $\mathbb{R}^{d'}$-valued processes sharing the properties $\mathscr{P}^*$. Denote by $\mathscr{C}_{in}$ the class of *competing models* for the input, that is, the collection of $\mathbb{R}^d$-valued stochastic processes consistent with the available information. For example, if the available

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0022-460X/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/S0022-460X(03)00645-X

information is limited to the first two moments of $\mathbf{X}$, then $\mathcal{C}_{in}$ is the class of $\mathbb{R}^d$-valued processes in L_2 with prescribed second-moment properties.

Let $\mathcal{P}^{**}$ denote the input properties needed to calculate the required output properties $\mathcal{P}^*$ for $\mathbf{Y}$. Denote by $\mathcal{H}^{**}$ the class of $\mathbb{R}^d$-valued stochastic processes sharing the properties $\mathcal{P}^{**}$. Generally, the determination of the properties $\mathcal{P}^*$ requires information on $\mathbf{X}$ beyond $\mathcal{P}^*$, that is, the properties $\mathcal{P}^{**}$ have to provide a more detailed description than the properties $\mathcal{P}^*$. For example, the calculation of the first two moments of the output $\mathbf{Y}$ of a non-linear dynamic system subjected to an input $\mathbf{X}$ requires information on $\mathbf{X}$ beyond its second-moment properties. If $\mathcal{C}_{in} \subseteq \mathcal{H}^{**}$, the information available on the input is sufficient to calculate the properties $\mathcal{P}^*$ for $\mathbf{Y}$ uniquely. Otherwise, the available information on the input is insufficient for finding the required output properties $\mathcal{P}^*$, so that this information has to be augmented to calculate $\mathcal{P}^*$. Let $\mathcal{C}_{in,a}$ denote a collection of $\mathbb{R}^d$-valued processes such that (1) the members of $\mathcal{C}_{in,a}$ are also in $\mathcal{C}_{in}$ and (2) the output properties $\mathcal{P}^*$ can be calculated if the input is represented by any of the members of $\mathcal{C}_{in,a}$. Generally, $\mathcal{P}^*$ depends on the particular member in $\mathcal{C}_{in,a}$ used to represent the input, so that the required properties of $\mathbf{Y}$ cannot be found uniquely. The above procedure of enhancing the available information on the input is common in applications, for example, seismic ground acceleration records are viewed as samples of a Gaussian process for dynamic analysis [1, Sections 3.1 and 3.2].

The main objective of the paper is the evaluation of the sensitivity of output properties to the input models in $\mathcal{C}_{in,a}$. It is shown that some output properties depend strongly on the particular member in $\mathcal{C}_{in,a}$ used to model the input. This dependence can have significant practical implications since any member of $\mathcal{C}_{in,a}$ is a valid input model. Simple linear/non-linear systems with quasi-static/dynamic outputs are used to illustrate the sensitivity of some output properties to the particular input model selected from $\mathcal{C}_{in,a}$.

The paper does not address the problem of selecting the optimal member of $\mathcal{C}_{in,a}$. A solution of this model selection problem is offered by the Bayesian decision method in Ref. [2]. The method accounts for both the available information and the consequence of using an inadequate input model. The states of nature and the action space of the decision framework in Ref. [2] coincide with the members $\mathcal{M}_i$, $i = 1, \ldots, m$, of $\mathcal{C}_{in,a}$. The utility function $u(\mathcal{M}_i, \mathcal{M}_j)$, $i, j = 1, \ldots, m$, defines the penalty of representing the input by $\mathcal{M}_i$ under the assumption that the nature state is $\mathcal{M}_j$. The optimal model minimizes the expected utility $\sum_{j=1}^{m} u(\mathcal{M}_i, \mathcal{M}_j)p_j$, where p_j denotes the probability that $\mathcal{M}_j$ is the actual state of nature.

2. Equivalent classes for stochastic processes

Several classes of equivalence are defined for stochastic processes corresponding to various levels of information on their probability law. Let $\mathcal{H}$ be the collection of $\mathbb{R}^d$-valued stochastic processes $\mathbf{X}(t)$, $t \in [t_0, t_1]$, defined on a probability space. If $\mathbf{X}$ is in L_2, its mean and correlation functions, $\boldsymbol{\mu}(t) = E[\mathbf{X}(t)]$ and $\mathbf{r}(t,s) = E[\mathbf{X}(t)\mathbf{X}(s)^{\mathrm{T}}]$, exist and are finite. We consider eight classes of equivalence for $\mathbb{R}^d$-valued stochastic processes:

- The class of processes equivalent in the *second-moment* sense,

$$\mathcal{H}_{sm} = \{\mathbf{X} \in \mathcal{H}: \ \mathbf{X} \in L_2 \text{ and } \mathbf{X} \sim (\boldsymbol{\mu}(t), \mathbf{r}(t,s))\}, \tag{1}$$

  consists of all processes in $\mathcal{H} \cap L_2$ with the same mean and correlation functions.

- The class of processes equivalent in the *higher order moment* sense,

$$\mathscr{H}_{hocm} = \{\mathbf{X}\in\mathscr{H}_{sm}\colon\ \mathbf{X}\in L_m, m\geqslant 3,\ \text{and}\ E[X_i(t)^k],\ k=3,\ldots,m,\ i=1,\ldots,d\}, \tag{2}$$

  is a subset of $\mathscr{H}_{sm}$ including processes in L_m, $m\geqslant 3$, whose co-ordinates $X_i(t)$, $i=1,\ldots,d$, have the same moments up to order $m\geqslant 3$ at each time $t\in[t_0,t_1]$.
- The class of processes equivalent in the *higher order correlation* sense,

$$\mathscr{H}_{hoc} = \left\{\mathbf{X}\in\mathscr{H}_{sm}\colon\ \mathbf{X}\in L_m, m\geqslant 3,\ \text{and}\ E\left[\prod_{q=1}^{m} X_{k_q}(u_q)\right],\ k_q\in\{1,2,\ldots,d\}\right\}, \tag{3}$$

  is a subset of $\mathscr{H}_{sm}$ including processes in L_m, $m\geqslant 3$, that have the same higher order correlation functions $E\left[\prod_{q=1}^{m} X_{k_q}(u_q)\right]$ up to order $m\geqslant 3$, where the indices $k_q\in\{1,2,\ldots,d\}$ do not have to be distinct.
- The class of processes equivalent in the *second-moment and co-ordinate distribution* sense,

$$\mathscr{H}_{sm,cd} = \{\mathbf{X}\in\mathscr{H}_{sm}\colon\ \text{same marginal distributions for co-ordinates}\}, \tag{4}$$

  is a subset of $\mathscr{H}_{sm}$ including processes whose co-ordinates X_i, $i=1,\ldots,d$, have the same marginal distributions.
- The class of processes equivalent in the *second-moment and distribution* sense,

$$\mathscr{H}_{sm,d} = \{\mathbf{X}\in\mathscr{H}_{sm}\colon\ \text{same marginal distributions}\}, \tag{5}$$

  is a subset of $\mathscr{H}_{sm}$ with the same marginal distributions. The classes $\mathscr{H}_{sm,cd}$ and $\mathscr{H}_{sm,d}$ coincide for real-valued processes ($d=1$).
- The class of processes with the same finite-dimensional distributions,

$$\mathscr{H}_{v} = \{\mathbf{X}\in\mathscr{H}\colon\ \text{same finite-dimensional distributions}\}, \tag{6}$$

  are called *versions*. Versions may or may not have finite moments.

The processes in the above classes of equivalence do not have to be defined on the same probability space. For example, consider two real-valued processes X and X' that are defined on the probability spaces $(\Omega,\mathscr{F},P)$ and $(\Omega',\mathscr{F}',P')$, respectively. These processes are versions if the probabilities $P(\cap_{q=1}^{n} X(u_q)\leqslant x_q)$ and $P'(\cap_{q=1}^{n} X'(u_q)\leqslant x_q)$ coincide for any integer $n\geqslant 1$, $u_q\in[t_0,t_1]$, $x_q\in\mathbb{R}$, and $q=1,\ldots,n$. The processes in the following two classes of equivalence have to be defined on the same probability space.

- The class of processes defined on a probability space $(\Omega,\mathscr{F},P)$ such that

$$\mathscr{H}_{m} = \{\mathbf{X},\mathbf{Y}\in\mathscr{H}_{v}\colon\ P(\omega\in\Omega\colon\ \mathbf{X}(t,\omega)=\mathbf{Y}(t,\omega)) = 1\ \text{at each}\ t\in[t_0,t_1]\}. \tag{7}$$

  The processes $\mathbf{X}$ and $\mathbf{Y}$ with the above property are called *modifications*.

- The class of processes defined on a probability space $(\Omega, \mathscr{F}, P)$ such that

$$\mathscr{H}_i = \{\mathbf{X}, \mathbf{Y} \in \mathscr{H}_v\text{: almost all samples of } \mathbf{X} \text{ and } \mathbf{Y} \text{ coincide}\}. \tag{8}$$

  The processes $\mathbf{X}$ and $\mathbf{Y}$ with the above property are said to be *indistinguishable*. Indistinguishable processes are modifications but the converse is not generally true [3, Section 3.8].

The sample properties for the stochastic processes in the above classes of equivalence have not been specified. These properties are needed to characterize fully a stochastic process. For example, the Brownian motion $B(t)$, $t \geqslant 0$, is by definition a real-valued process with increments independent of the past such that $B(t) - B(s) \sim N(0, t-s)$, $s < t$, where $N(\mu, \sigma^2)$ denotes a Gaussian random variable with mean μ and variance σ^2. This definition yields the finite-dimensional distributions of B, that is, it specifies B as a member in $\mathscr{H}_v$, but does not tell anything about the samples of this process. It can be shown that there is a modification of B which has continuous samples a.s. [4, Theorem 26, p. 17]. This modification of B is commonly referred to as the Brownian motion process, and continues to be denoted by B. We also note that B so defined is a member on $\mathscr{H}_i$ since modifications with right or left continuous samples are indistinguishable [3, Example 3.34, p. 138].

Example 1. Let $B(t)$, $t \geqslant 0$, be a Brownian motion process and let

$$C(t) = \sum_{k=1}^{N(t)} Y_k = \sum_{k=1}^{\infty} Y_k 1(t \geqslant T_k), \quad t \geqslant 0, \tag{9}$$

be a compound Poisson process, where N denotes a homogeneous Poisson process with intensity $\lambda > 0$; Y_k, $k = 1, 2, \ldots$, are independent identically distributed real-valued random variables; T_k, $k = 1, 2, \ldots$, denote the jump times of N, and $T_0 = 0$. The process C has right continuous samples by definition. The second-moment properties of the Brownian motion process are $E[B(t)] = 0$ and $E[B(t)B(s)] = t \wedge s$, where $t \wedge s = \min(t, s)$. If Y_1 is in L_2 such that $E[Y_1] = 0$ and $\lambda E[Y_1^2] = 1$, then C has the same second-moment properties as B so that $B, C \in \mathscr{H}_{sm}$ with $d = 1$, $\mu(t) = 0$, and $r(t, s) = t \wedge s$. Also, the formal derivatives of B and C are the Gaussian and the Poisson white-noise processes, respectively. These white noise processes are equivalent in the second-moment sense.

Suppose that Y_1 satisfies the above conditions, that is, $E[Y_1] = 0$ and $\lambda E[Y_1^2] = 1$, so that $B, C \in \mathscr{H}_{sm}$. The processes B and C have some other common properties besides their first two moments. For example, B and C (1) have stationary independent increments so that they are Markov, (2) are not m.s. differentiable since the correlation function $r(t, s)$ is not differentiable at $t = s$, and (3) are continuous in probability and mean square. However, there are notable differences between these processes. The Brownian motion has a.s. continuous samples, while the samples of C have jumps. The random variable $B(t)$ is Gaussian with mean 0 and variance t. The random variable $C(t)$ is not Gaussian, and its distribution depends on λ, the distribution of Y_1, and time of t [3, Sections 3.12 and 3.13]. Under the above assumptions on Y_1, the sequence of random variables $C(t)/\sqrt{t}$ converges in distribution to $N(0, 1)$ as $t \to \infty$ by the central limit theorem [5, Theorem 9.7.1, p. 313] since $N(t)/t$ converges a.s. to λ as $t \to \infty$ [6, Theorem 3.3.2,

p. 189]. Hence, $C(t)$ can be approximated by a Gaussian variable with mean 0 and variance t for large times. The coefficient of skewness and kurtosis of $B(t)$ are $\gamma_{3,B}(t) = 0$ and $\gamma_{4,B}(t) = 3$, respectively. The corresponding coefficients of $C(t)$ are $\gamma_{3,C}(t) = 0$ and

$$\gamma_{4,C}(t) = 3 + \frac{\lambda E[Y_1^4]}{t} \tag{10}$$

provided that Y_1 has also the properties $E[Y_1^3] = 0$ and $E[Y_1^4] < \infty$. The above formula for $\gamma_{4,C}$ results from the expression of the cumulants of filtered Poisson processes [7, Section 3.3] and the relationship between cumulants and moments [7, Appendix B]. Fig. 1 shows 20 samples of the Brownian motion B and of a compound Poisson process C with $\lambda = 0.05, 0.1, 0.5$ and $Y_1 \sim N(0, 1/\lambda)$. The differences between the samples of B and C are significant for small values of λ but decrease with λ, an expected result since C with the above properties approaches a Brownian motion as $\lambda \to \infty$ [7, Example 3.13, p. 85].

Example 2. Consider the processes $X_T(t)$, $X_D(t)$, and $X_P(t)$, $t \geqslant 0$, defined by (1) the formula

$$X_T(t) = F^{-1} \circ \Phi(G(t)), \tag{11}$$

where F is a distribution, Φ denotes the distribution of $N(0, 1)$, and $G(t)$ is a stationary Gaussian process with mean 0, variance 1, and correlation function $\rho(\tau) = E[G(t+\tau)G(t)]$, (2) the stochastic

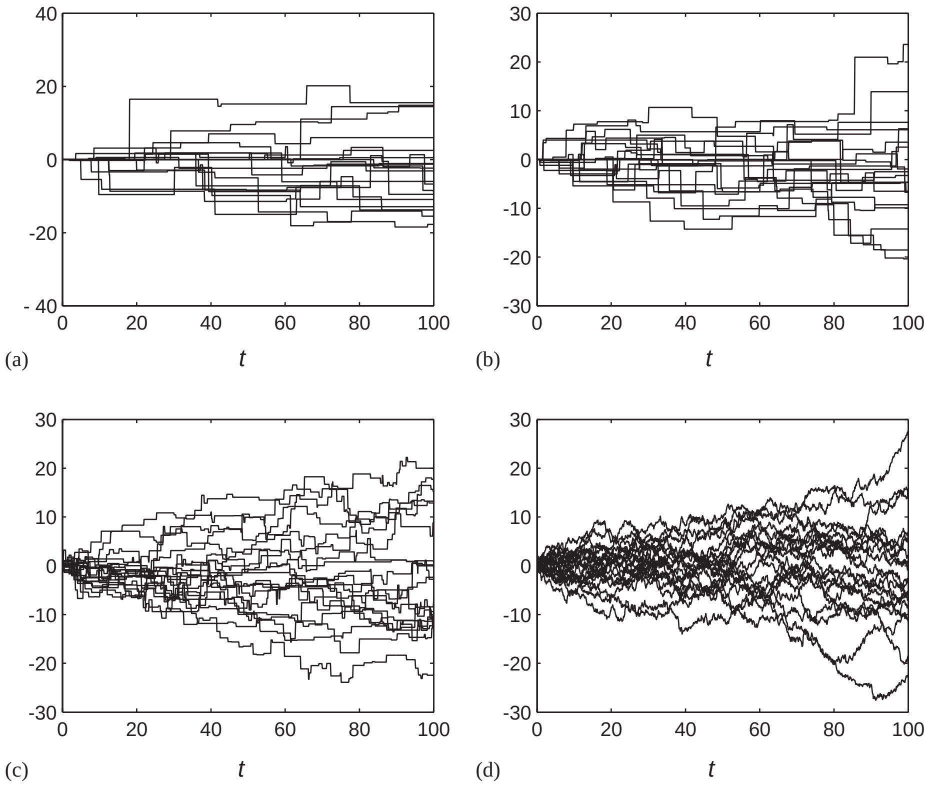

Fig. 1. Samples of C with (a) $\lambda = 0.05$, (b) $\lambda = 0.1$, and (c) $\lambda = 0.5$, and (d) a sample of B.

differential equation

$$\mathrm{d}X_D(t) = -\alpha X_D(t)\,\mathrm{d}t + b(X_D(t))\,\mathrm{d}B(t), \tag{12}$$

where b is a function satisfying the uniform Lipschitz conditions [3, Section 4.7.1.1] and B denotes a Brownian motion process; and (3) the summation

$$X_P(t) = \sum_{k=1}^{N(t)} Y_k h(t - T_k), \tag{13}$$

where N is a Poisson process with intensity $\lambda > 0$, $T_1, T_2, \ldots$ are the jump times of N, $Y_1, Y_2, \ldots$ denote independent identically distributed random variables in L_2, and h is a function such that $h(s) = 0$ for $s < 0$. The stationary cumulants of order $p = 1, 2, \ldots$ of $X_P(t)$ are $\chi_p = \lambda E[Y_1^p]/(p\alpha)$ [7, Section 3.3].

Let F_0 be the lognormal distribution

$$F_0(x) = \Phi\left(\frac{1}{\sigma}\ln(x\sqrt{\mathrm{e}^{2\sigma^2} - \mathrm{e}^{\sigma^2}} + \mathrm{e}^{\sigma^2/2})\right), \quad x > -\mathrm{e}^{\sigma^2/2}/\sqrt{\mathrm{e}^{2\sigma^2} - \mathrm{e}^{\sigma^2}}, \tag{14}$$

with mean 0 and variance 1, where $\sigma > 0$ is a scale parameter, and consider the exponential correlation function

$$\xi_0(\tau) = \exp(-\alpha\,|\tau|), \quad \alpha > 0. \tag{15}$$

If $F = F_0$ in Eq. (11), the translation process X_T has the marginal distribution F_0. The relationship between the correlation functions ξ_0 of X_T and the corresponding correlation function ρ_0 of G is [7, Eq. (3.33), p. 52]

$$\xi_0(\tau) = \frac{1 - \mathrm{e}^{\sigma^2\,\rho_0(\tau)}}{1 - \mathrm{e}^{\sigma^2}} \tag{16}$$

so that

$$\rho_0(\tau) = \frac{1}{\sigma^2}\ln\left(1 + (\mathrm{e}^{\sigma^2} - 1)\xi_0(\tau)\right). \tag{17}$$

The function ρ_0 is a legitimate correlation function since $\xi_0(\tau) \geqslant 0$ for $\tau \in \mathbb{R}$ by definition (Eq. (15)), $\xi_0(\tau) = 0$ if and only if $\rho_0(\tau) = 0$, $\xi_0(\tau) = 1$ if and only if $\rho_0(\tau) = 1$, $\xi_0(\tau)$ is an increasing function of $\rho_0(\tau)$ [7, Section 3.1.1], and

$$\sum_{i,j=1}^{n} a_i a_j \rho_0(t_i - t_j) \geqslant \left(\sum_{i,j=1}^{n} a_i a_j\right) \min_{1 \leqslant i,j \leqslant n} \rho_0(t_i - t_j) \geqslant 0$$

for any integer $n \geqslant 1$, times t_i, and constants a_i, so that ρ_0 is positive definite. The stationary solution X_D of Eq. (12) has the correlation function ξ_0 for any functional form of b. If

$$b(x)^2 = \frac{-2\alpha}{f_0(x)} \int_{x_l}^{x} u f_0(u)\,\mathrm{d}u, \tag{18}$$

then the marginal density of the stationary diffusion process X_D is $f_0(x) = \mathrm{d}F_0(x)/\mathrm{d}x$, where $x_l = \inf\{x \in \mathbb{R} : F_0(x) > 0\}$ [8]. The filtered Poisson process X_P in Eq. (13) with $E[Y_1] = 0$, $E[Y_1^2] = 2\,\alpha/\lambda$, and $h(s) = \mathrm{e}^{-\alpha s}$ for $s \geqslant 0$ has mean 0, variance 1, and the stationary correlation function ξ_0 by properties of filtered Poisson processes [7, Section 3.3]. Generally, X_P cannot match an arbitrary marginal distribution F_0. The parameter λ and the properties of Y_1 can be tuned such that, for example, the marginal distribution of X_P matches some higher order moments of F_0 provided they exist and are finite.

The processes X_T, X_D, and X_P corresponding to F_0 in Eq. (14) and ξ_0 in Eq. (15) have the following properties:

1. The processes X_T and X_D are equivalent in the sense of the class $\mathscr{H}_{sm,d}$ while X_P is equivalent to X_T and X_D only in the second-moment sense. Fig. 2 shows histograms of samples of the translation, diffusion, and filtered Poisson processes calibrated to the target statistics F_0 with $\sigma = 1$ and ξ_0 with $\alpha = 1$. The target density f_0 is also shown in all plots with continuous lines. The histograms of X_T and X_D are similar and match f_0 satisfactorily. On the other hand, the histogram of X_P corresponding to a lognormal variable Y_1 with mean 0, variance 1, and scale $\sigma = 1.1$ is at variance with f_0. For this choice of Y_1 the moments of order 3 and 4 of $X_P(t)$ are close to the corresponding moments of the target distribution F_0. Alternative distributions can be selected for Y_1; for example, mixtures of distributions may provide superior approximations for f_0. Generally, the marginal distribution of $X_P(t)$ cannot match exactly an arbitrary distribution F_0 regardless of the distribution selected for Y_1. Let $\varphi_0 = \int_{\mathbb{R}} e^{iux} f_0(x)\, dx$ denote the target characteristic function. It can be shown that the stationary characteristic function of X_P satisfies the equation $-\alpha u \varphi'_P(u) + \lambda(\varphi_{Y_1}(u) - 1)\varphi_P(u) = 0$, where φ_{Y_1} denotes the characteristic function of Y_1 [3, Example 7.34]. If the solution φ_{Y_1} of this equation with $\varphi_P = \varphi_0$ exists and is a characteristic function, then the stationary marginal distribution of X_P is F_0.

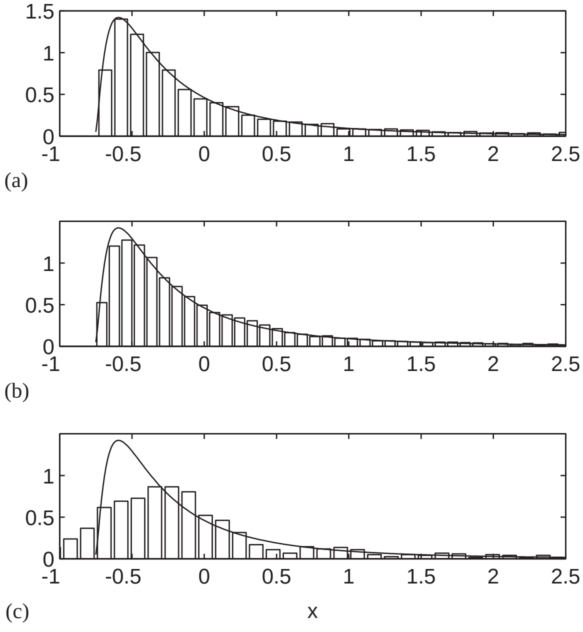

Fig. 2. Histograms of the lognormal (a) translation, (b) diffusion, and (c) filtered Poisson processes.

2. The processes X_T, X_D, and X_P are m.s. continuous but are not differentiable in the mean square sense since their correlation function ξ_0 is continuous but is not differentiable at $\tau = 0$ [3, Sections 3.9.1 and 3.9.2].
3. The processes X_T and X_D have a.s. continuous samples, while the samples of X_P exhibit jumps. That X_D has continuous samples follows from a property of diffusion processes [3, Section 4.7.1]. Since the mapping in Eq. (11) is continuous for $F = F_0$, the translation process X_T has continuous samples if and only if its Gaussian image G has continuous samples. We have
$$E[(G(t+h) - G(t))^4] = 12(1 - \rho_0(\tau))^2 = 12\left(1 - \frac{1}{\sigma^2}\ln(1 + (e^{\sigma^2} - 1)e^{-\alpha\,|h|})\right)^2$$
$$= 12g(h)^2$$
for any $h \in \mathbb{R}$ by properties of Gaussian variables and Eq. (17). Elementary calculations show that $0 \leqslant g(h) \leqslant g'(0+)\,|h|$, where $g'(0+) = \alpha(1 - e^{-\sigma^2})/\sigma^2$. Hence,
$$E[(G(t+h) - G(t))^4] = 12g(h)^2 \leqslant 12g'(0+)^2\,|h|^2$$
so that G has continuous samples a.s. by a Kolmogorov criterion [3, Section 3.3]. The process X_P is continuous in probability since the probability of the event $|X_P(t) - X_P(s)| > \varepsilon$ converges to 0 for any $\varepsilon > 0$ as $|t - s| \to 0$.
4. Since X_D is a diffusion process, it is also Markov [3, Section 4.7.1.1]. Also, X_P is a Markov process because it can be viewed as the solution of the stochastic differential equation $dX_P(t) = -\alpha\, X_P(t)\,dt + dC(t)$, where C is a compound Poisson process (Eq. (9)). Generally, X_T does not have the Markov property. Since X_T is a memoryless transformation of G (Eq. (11)), X_T is a Markov process if and only if G has the Markov property. The necessary and sufficient condition for a Gaussian process G to be Markov is that $\rho_0(t-u) = \rho_0(t-s)\rho_0(s-u)$ holds for every $t > s > u$ [9, Section 2.5]. This condition is not satisfied at all times; for example, $\rho_0(t-u) = 0.4899$ and $\rho_0(t-s)\rho_0(s-u) = 0.5098$ for $t = 1$, $s = 0.5$, $u = 0$, the exponential correlation in Eq. (15) with $\alpha = 1$, and the marginal distribution in Eq. (14) with $\sigma = 1$.
5. The conditional distributions of X_D and X_T can differ significantly. For example, the joint distribution of X_T at two arbitrary times s and t, $s < t$, is
$$P(X_T(s) \leqslant x_1, X_T(t) \leqslant x_2) = P(G(s) \leqslant y_1, G(t) \leqslant y_2) = \Phi(y_1, y_2; \rho_0(\tau)), \tag{19}$$
where $\tau = t - s$ denotes the time lag, $\Phi(y_1, y_2; \rho_0(\tau))$ is the density of the Gaussian vector $(G(s), G(t))$ and $y_i = \log(x_i\sqrt{e^{2\sigma^2} - e^{\sigma^2}} + e^{\sigma^2/2})/\sigma$. The joint density of $(X_T(s), X_T(t))$ results by differentiating Eq. (19). The ratio of this density to the density of the random variable $X_T(s)$ gives the density of the conditional variable $X_T(t) \mid X_T(s)$. The density of $X_D(t) \mid X_D(s)$ cannot be found analytically, but can be obtained numerically by, for example, the path integral method. The method is based on the observation that a diffusion process is locally Gaussian, that is, the conditional variable $X_D(t + \Delta t) \mid X_D(t) = x$, $\Delta t > 0$, is approximately Gaussian with mean $(1 - \alpha\Delta t)x$ and variance $b(x)^2\Delta t$ for small values of the time step Δt [3, Section 7.3.1.5]. Fig. 3 shows conditional densities for the lognormal translation and diffusion processes considered in Fig. 2 for $\alpha = 1$, $\sigma = 1$, and two time lags, $\tau = 1$ and $\tau = 10$. Although these processes are equivalent in the sense of the class $\mathscr{H}_{sm,d}$, their conditional densities differ significantly. The translation process has a shorter memory than the diffusion model in the sense that its conditional density for $\tau = 10$ is nearly equal with the marginal density f_0, while the

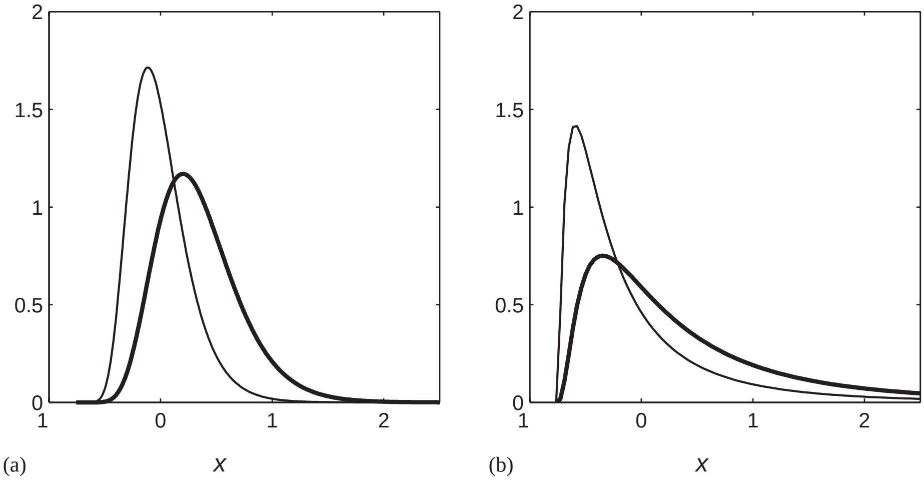

Fig. 3. Conditional densities of a lognormal (a) diffusion process and (b) translation process. The heavy and thin lines are for $\tau = 1$ and $\tau = 10$.

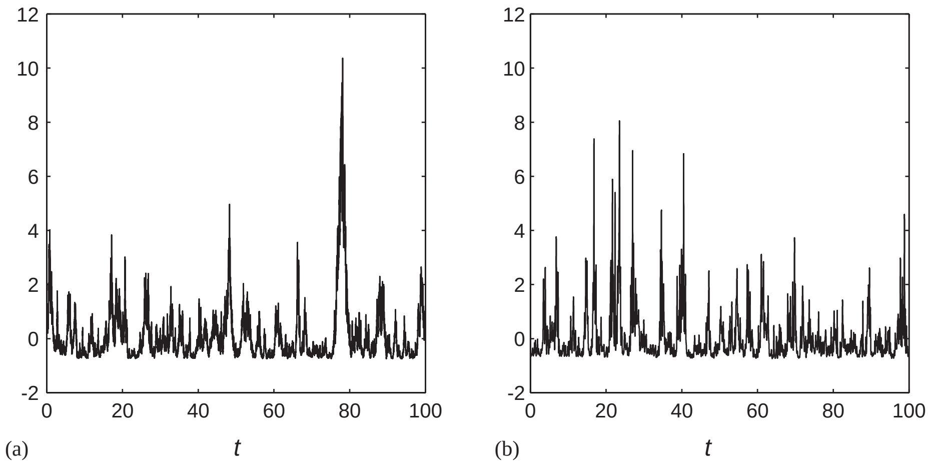

Fig. 4. Samples of the (a) diffusion process X_D and (b) translation process X_T.

corresponding conditional density of the diffusion process for $\tau = 10$ differs from f_0. The plots in Fig. 3 also show that X_T and X_D are not versions since their finite-dimensional distributions differ. Fig. 4 shows samples of the lognormal diffusion and translation processes considered in this example. It is not possible to infer from these samples by visual inspection the significant differences between the conditional densities of X_D and X_T illustrated in Fig. 3.

The above properties show that the processes X_T, X_D, and X_P are valid models for a real-valued stochastic process X with mean 0 and correlation function in Eq. (15). If in addition it is known that X is a stationary process with the marginal distribution F_0 in Eq. (14), the processes X_T and

X_D are possible models for X. Some of the properties of the above models for X differ significantly. And the processes X_T, X_D, and X_P are just a few members of a relatively large class of competing models for X.

Example 3. Consider the probability space $(\Omega = [0,1], \mathscr{F} = \mathscr{B}([0,1]), P(\mathrm{d}\omega) = \mathrm{d}\omega)$, where $\mathscr{B}([0,1])$ denotes the Borel σ-field on $[0,1]$. Let $X(t,\omega) = 0$ for all (t,ω) and $Y(t,\omega) = 0$ and 1 for $t \neq \omega$ and $t = \omega$, respectively, be two stochastic processes defined on $(\Omega, \mathscr{F}, P)$. These processes are versions and modifications since $\Omega_t = \{\omega \in \Omega : X(t,\omega) \neq Y(t,\omega)\} = \{t\}$ and $P(\{t\}) = 0$ for $t \in [0,1]$. However, X and Y are not indistinguishable processes since they have different samples [3, Example 3.35, p. 138]. For example, the maxima of these processes are $\max_{t \in [0,1]} X(t) = 0$ and $\max_{t \in [0,1]} Y(t) = 1$ a.s.

The differences between the samples of modifications are not relevant when the output is a weighted integral of past input values, for example, the response of a linear system to a random input. In this case, the differences between the samples of various input models disappear in the output, as it will be seen later in Example 5.

3. Response to equivalent models

Suppose that the objective is to find some properties $\mathscr{P}^*$ defining a class of equivalence $\mathscr{H}^*$ for the output, for example, the second-moment properties $\mathscr{P}^*$ defining the class of processes $\mathscr{H}^* = \mathscr{H}_{sm}$. Generally, we need to know the input beyond the properties $\mathscr{P}^*$ to find the properties $\mathscr{P}^*$ for the output. Let $\mathscr{P}^{**}$ denote the input properties needed to find the properties $\mathscr{P}^*$ for the output. Denote by $\mathscr{H}^{**}$ the class of $\mathbb{R}^d$-valued stochastic processes with the properties $\mathscr{P}^{**}$. The relationship between $\mathscr{P}^*$ and $\mathscr{P}^{**}$ depends largely on the features of the system response, for example, linear/non-linear and quasi-static/dynamic response.

Let $\mathscr{C}_{in}$ denote the collection of $\mathbb{R}^d$-valued processes that are consistent with the available information on the input, that is, the class of competing models. If $\mathscr{C}_{in} \subseteq \mathscr{H}^{**}$, the available information on the input suffices to find the properties $\mathscr{P}^*$ uniquely. In this case the properties $\mathscr{P}^*$ depend only on the properties $\mathscr{P}^{**}$ of the input. For example, the second-moment properties of the response of a linear system can be calculated from the second-moment properties of the input. The particular member in $\mathscr{C}_{in}$ used to represent the input is irrelevant. However, properties of the output beyond $\mathscr{P}^*$ may depend on the member in $\mathscr{C}_{in}$ used to represent the input. If $\mathscr{C}_{in} \supset \mathscr{H}^{**}$, the available information on the input is insufficient to calculate the output properties $\mathscr{P}^*$. The properties of the members in $\mathscr{C}_{in}$ need to be enhanced to characterize the output at the required level. Let $\mathscr{C}_{in,a}$ be a collection of processes satisfying two conditions. The processes in $\mathscr{C}_{in,a}$ are also members of $\mathscr{C}_{in}$ and their definition provides adequate information for calculating the properties $\mathscr{P}^*$ for $\mathbf{Y}$. Generally, the properties $\mathscr{P}^*$ depend on the particular member in $\mathscr{C}_{in,a}$ used to represent the input, so that it is not possible to find these properties uniquely. It is necessary to represent the input $\mathbf{X}$ by a single member in $\mathscr{C}_{in,a}$ to find $\mathscr{P}^*$ uniquely. As previously stated, an optimal model for $\mathbf{X}$ can be extracted from $\mathscr{C}_{in,a}$ by, for example, the Bayesian decision method in Ref. [4].

3.1. Linear systems

The quasi-static and dynamic system responses are examined separately since their dependence on input differs in an essential way. A quasi-static response is a memoryless linear mapping of input. On the other hand, a dynamic response $\mathbf{Y}(t)$ at an arbitrary time t depends on the entire input history $\{\mathbf{X}(s),\ s \leqslant t\}$ and the initial value of $\mathbf{Y}$.

3.1.1. Quasi-static response

Let $\mathbf{Y}(t) = \mathbf{a}\mathbf{X}(t)$ be the system response or output, where $\mathbf{a}$ denotes a deterministic matrix depending on the system properties.

If the second-moment properties of $\mathbf{Y}$ are required, it is sufficient to describe $\mathbf{X}$ as a member of the class of equivalence $\mathscr{H}_{sm}$. In this case, the required output properties $\mathscr{P}^*$ and the input properties $\mathscr{P}^{**}$ needed to delivered the output characterization coincide, and consists of the second-moment properties of these processes. If the information on $\mathbf{X}$ is increased so that it becomes a member of the classes of equivalence $\mathscr{H}_{hocm}$ or $\mathscr{H}_{sm,cd}$, $d > 1$, then $\mathbf{Y}$ cannot be characterized beyond its second-moment properties. The additional information on the input cannot be transferred to the output. For example, let $d = 2$, $d' = 1$, and $Y(t) = X_1(t) + X_2(t)$. The third moment of $Y(t)$ cannot be calculated if $\mathbf{X} \in \mathscr{H}_{hocm}$ because the moments $E[X_1(t)^2 X_2(t)]$ and $E[X_1(t) X_2(t)^2]$ are not known. Hence, information on the input beyond the defining properties of the classes $\mathscr{H}_{hocm}$ and $\mathscr{H}_{sm,cd}$ is needed to calculate the properties of Y corresponding to these classes.

If $\mathbf{X}$ is a member of $\mathscr{H}_{hoc}$, $\mathscr{H}_{sm,d}$, $\mathscr{H}_v$, $\mathscr{H}_m$, or $\mathscr{H}_i$, the output $\mathbf{Y}$ can be characterized at the same level as the input, that is, the properties $\mathscr{P}^*$ and $\mathscr{P}^{**}$ coincide. For example, the finite-dimensional distributions for $\mathbf{Y}$ can be obtained from the corresponding family of distributions for $\mathbf{X}$.

3.1.2. Dynamic response

If $\mathbf{X}$ is in $\mathscr{H}_{sm}$, the second-moment properties of the output $\mathbf{Y}$ can be obtained by classical methods of linear random vibration [3, Section 7.2.1.2]. Hence, it is sufficient to specify the second-moment properties of the input to calculate the same output properties. Adding information on $\mathbf{X}$ so that it becomes a member of $\mathscr{H}_{hocm}$, $\mathscr{H}_{sm,cd}$, or $\mathscr{H}_{sm,d}$ does not allow the determination of statistics of $\mathbf{Y}$ beyond its first two moments. For example, let $Y(t) = \int_{t_0}^{t} h(t,s)\, X(s)\, \mathrm{d}s$ be the output of a system with $d = d' = 1$, the Green function $h(\cdot\,,\cdot)$, and initial value $Y(t_0) = 0$. The distribution of the random variable $Y(t)$ depends on the entire history of X in the time interval $[t_0, t]$, so that it cannot be calculated if X is a member of $\mathscr{H}_{sm,cd}$ or $\mathscr{H}_{sm,d}$. It is necessary to describe X in much more detail to find the marginal distribution of Y, that is, the properties $\mathscr{P}^{**}$ must provide a much more refined characterization for X than the required properties $\mathscr{P}^*$ for Y.

If $\mathbf{X}$ is a member of $\mathscr{H}_{hoc}$, $\mathscr{H}_v$, $\mathscr{H}_m$, and $\mathscr{H}_i$, the input–output mapping preserves the information on the input, that is, it delivers properties $\mathscr{P}^*$ for $\mathbf{Y}$ compatible with these classes of equivalence. For example, higher order correlations of the output $Y(t) = \int_{t_0}^{t} h(t,s)\, X(s)\, \mathrm{d}s$ considered above can be obtained from the corresponding correlations of the input, which are available for $X \in \mathscr{H}_{hoc}$.

Example 4. Let $Y(t)$, $t \geqslant 0$, be the solution of the stochastic differential equation

$$\mathrm{d}Y(t) = -\beta Y(t)\, \mathrm{d}t + \sqrt{2\beta}\, \mathrm{d}X(t), \quad t \geqslant 0, \tag{20}$$

with the initial condition $Y(0) = 0$, where $\beta > 0$ is a constant. It is assumed that X is in L_2 and we only know the mean $E[X(t)] = 0$ and correlation function $E[X(t)X(s)] = t \wedge s$ of this process, so that $\mathscr{C}_{in} = \mathscr{H}_{sm}$. The second-moment properties of the output Y are

$$\mu(t) = E[Y(t)] = 0,$$

$$r(t,s) = E[Y(t)Y(s)] = (1 - \mathrm{e}^{-2\beta(t \wedge s)})\,\mathrm{e}^{-\beta\,|t-s|}, \tag{21}$$

by classical methods of linear random vibration [3, Section 7.2.1.2]. Hence, the first two moments of Y can be calculated from the corresponding input moments so that $\mathscr{P}^{**} = \mathscr{P}^{*}$.

Suppose now that (1) higher order moments and other properties of the solution Y of Eq. (20) are required and (2) the available information on the input X consists as above of its first two moments, that is, $\mathscr{C}_{in} = \mathscr{H}_{sm}$. This information is insufficient for finding the required output properties. For a solution we need to represent X by members in $\mathscr{C}_{in,a}$. Let B be a Brownian motion process and C a compound Poisson process (Eq. (9)) such that $Y_1 \in L_2$, $E[Y_1] = 0$, and $\lambda E[Y_1^2] = 1$. These processes have the specified input second-moment properties, so that they are in the class of competing models for X, that is, $B, C \in \mathscr{C}_{in}$. Also, B and C are adequately defined to allow the calculation of the required output properties, that is, $B, C \in \mathscr{C}_{in,a}$. Let Y_B and Y_C denote the solutions of Eq. (20) for $X = B$ and $X = C$, respectively. The processes Y_B and Y_C are equivalent in the second-moment sense, that is, they are members of $\mathscr{H}_{sm}$ with the first two moments in Eq. (21). However, their statistics beyond the second-moment properties differ significantly. For example, Y_B is a Gaussian process with continuous samples a.s., while Y_C is not Gaussian and its samples have jumps. An alternative form of Y_C is

$$Y_C(t) = \sqrt{2\beta} \sum_{k=1}^{N(t)} Y_k \mathrm{e}^{-\beta\,(t-T_k)}, \tag{22}$$

where T_k, $k \geqslant 1$, denote the jump times of C. If Y_1 is in L_p, the cumulants of order p of the random variable $Y_C(t)$ are [7, p. 295]

$$\chi_p(t) = \frac{\lambda E[Y_1^p](2\beta)^{p/2}}{p\beta}(1 - \mathrm{e}^{-p\beta t}) \tag{23}$$

so that the skewness and kurtosis coefficients of $Y_C(t)$ are

$$\gamma_3(t) = 0 \quad \text{and} \quad \gamma_4(t) = 3 + \frac{\beta\gamma_{Y_1,4}}{\lambda}\frac{1 - \mathrm{e}^{-4\beta t}}{(1 - \mathrm{e}^{-2\beta t})^2} \tag{24}$$

for $E[Y_1^3] = 0$, where $\gamma_{Y_1,4}$ denotes the kurtosis coefficient of Y_1. Since β, λ, and $\gamma_{Y_1,4}$ are positive, we have $\gamma_4(t) \geqslant 3$ at all times $t \geqslant 0$ so that the distribution of $Y_C(t)$ has a heavier tail than the distribution of $Y_B(t)$. The kurtosis coefficient $\gamma_4(t)$ approaches the value 3 corresponding to Gaussian variables as $\lambda \to \infty$. Fig. 5 shows histograms of Y_C corresponding to a compound Poisson process C with $Y_1 \sim N(0, 1/\lambda)$ for three values of λ and a histogram of Y_B. The histograms are based on 500 samples and correspond to $\beta = 1$. The differences between the marginal distributions of Y_B and Y_C are significant for small values of λ. These differences decrease as λ increases, consistently with the asymptotic behavior of $\gamma_4(t)$. Fig. 6 shows estimates of the probabilities $P(\max_{0 \leqslant t \leqslant \tau} |Y_B(t)| > y)$ and $P(\max_{0 \leqslant t \leqslant \tau} |Y_C(t)| > y)$ obtained from 500

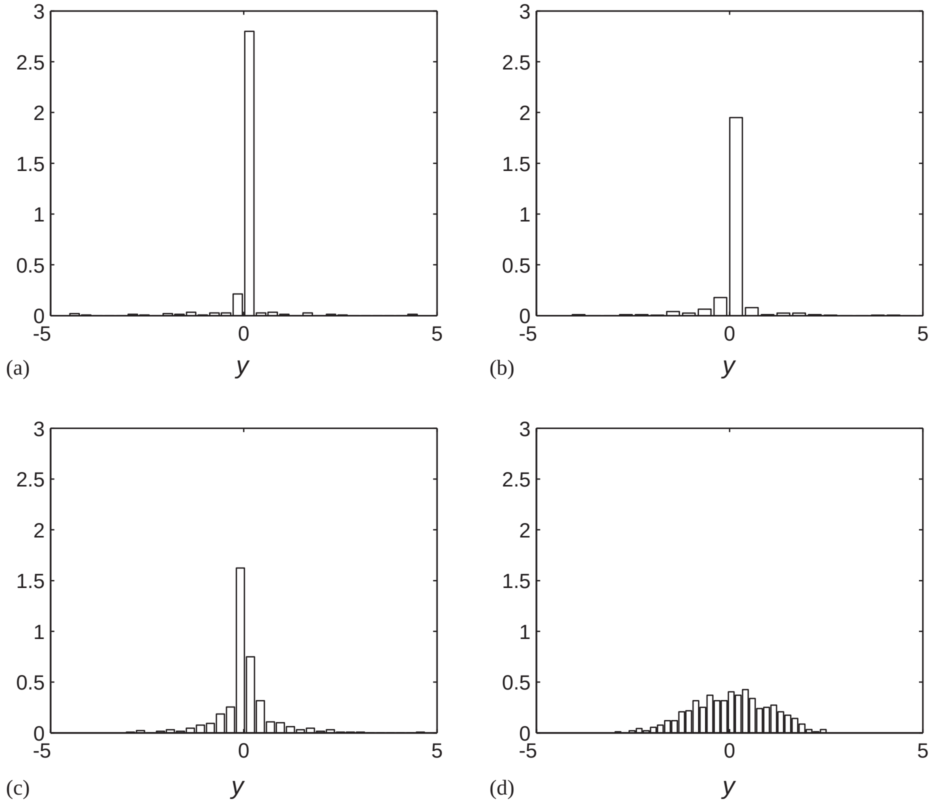

Fig. 5. Histograms of Y_C with $Y_1 \sim N(0, 1/\lambda)$, $\beta = 1$, and (a) $\lambda = 0.05$, (b) $\lambda = 0.1$, (c) $\lambda = 0.5$, and (d) a sample of Y_B.

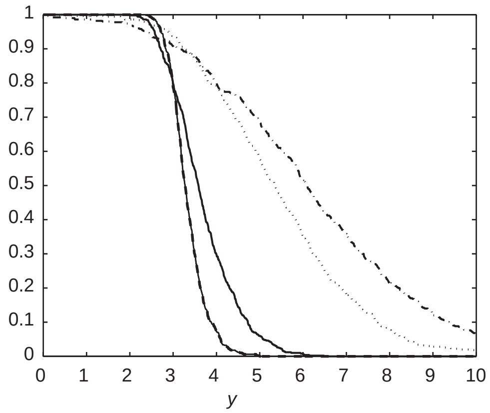

Fig. 6. Probabilities $P(\max_{0 \leqslant t \leqslant \tau} |Y_B(t)| > y)$ (dashed line) and $P(\max_{0 \leqslant t \leqslant \tau} |Y_C(t)| > y)$ for $\beta = 1$, and $\lambda = 0.5$ (solid line), $\lambda = 0.1$ (dotted line), and $\lambda = 0.01$ (dash-dotted line). All probabilities are for $\tau = 100$.

samples of Y_B and Y_C for $\beta = 1$, $\tau = 100$, and three values of λ. The upper tail of the probability $P(\max_{0 \leqslant t \leqslant \tau} |Y_C(t)| > y)$ dominates the upper tail of $P(\max_{0 \leqslant t \leqslant \tau} |Y_B(t)| > y)$ for small values of λ, but these differences decrease as λ increases in agreement with results in Fig. 5.

The probabilities in Fig. 6 can be interpreted as probabilities of failures during a time interval $[0, \tau]$.

The above results show that it is not possible to characterize uniquely the output beyond its first two moments if the information on the input is limited to its second-moment properties. The information on the input needs to be augmented to find the required output properties. Since the information on the input can be enhanced in many ways, the resulting output properties cannot be obtained uniquely, as demonstrated by the plots in Figs. 5 and 6. The lack of uniqueness of some properties of Y can have notable practical implications. For example, the probability of failure $P(\max_{0 \leqslant t \leqslant \tau} |Y(t)| > y)$ can be severely underestimated if it is selected arbitrarily to represent the input by $X = B$ and the actual input is $X = C$.

Example 5. Consider the probability space in Example 3 and the input processes $X_1(t)$ and $X_2(t, \omega) = X_1(t, \omega)$ for $t \neq \omega$ and $X_2(t, \omega) = X_1(t, \omega) + 1$ for $t = \omega$. The processes X_1 and X_2 are modifications but are not indistinguishable. Consider a linear system with transfer function $h(\cdot, \cdot)$ such that $h(t, s) = 0$ for $s > t$. The responses of this system to the inputs X_1 and X_2 are

$$Y_1(t, \omega) = \int_0^t h(t, s) X_1(s, \omega)\, ds,$$
$$Y_2(t, \omega) = \int_0^t h(t, s)[X_1(s, \omega) + 1(s = \omega)]\, ds = Y_1(t, \omega) \tag{25}$$

for zero initial condition, respectively. The last equality holds since the Lebesgue measure of the set $\{\omega\}$ is zero. The output processes Y_1 and Y_2 are not only modifications but are also indistinguishable, showing that some differences in the input sample properties may not be present in the output samples.

3.2. Non-linear systems

As for linear systems we examine the quasi-static and dynamic responses separately. A quasi-static response constitutes a memoryless non-linear mapping of input, while a dynamic response depends on the input history till the current time.

3.2.1. Quasi-static response

If the input $\mathbf{X}$ is a member of $\mathscr{H}_{sm}$, $\mathscr{H}_{hocm}$, $\mathscr{H}_{sm,cd}$, or $\mathscr{H}_{sm,d}$, it is not possible to calculate even the second-moment properties of the output $\mathbf{Y}$. For example, let $d = 2$, $d' = 1$, and $Y(t) = (X_1(t) + X_2(t))^2$ be a quasi-static response. If $\mathbf{X} \in \mathscr{H}_{sm}$, then $E[Y(t)^2]$ cannot be calculated since it involves moments of $\mathbf{X}(t)$ of order larger than 2. Also, $E[Y(t)^2]$ cannot be found if $\mathbf{X} \in \mathscr{H}_{hocm}$ or $\mathbf{X} \in \mathscr{H}_{sm,cd}$ since the expectations $E[X_1(t)^p X_2(t)^{4-p}]$ cannot be calculated for $p = 1, 2, 3$. If $\mathbf{X} \in \mathscr{H}_{sm,d}$, it is possible to find all marginal properties of the stochastic process Y, but the correlation function $E[Y(t)Y(s)]$ of this process cannot be obtained. Although the defining properties $\mathscr{P}^{**}$ for $\mathbf{X}$ are well beyond its first two moments, it is not possible to find the output second-moment properties.

On the other hand, properties $\mathscr{P}^{*}$ of $\mathbf{Y}$ at levels corresponding to the classes of equivalence $\mathscr{H}_v$, $\mathscr{H}_m$, and $\mathscr{H}_i$ can be obtained if the input $\mathbf{X}$ is in $\mathscr{H}_v$, $\mathscr{H}_m$, and $\mathscr{H}_i$, respectively. Also, if $\mathbf{X}$ is a member of $\mathscr{H}_{hoc}$ and the input–output mapping is polynomial, it is possible to find correlations of $\mathbf{Y}$ up to an order equal to the order p_c of the highest known correlation of $\mathbf{X}$ less the degree p_r of the input–output polynomial relationship provided that $p_c \geqslant p_r$. For example, we can find correlations up to order 3 for the output process $Y(t) = (X_1(t) + X_2(t))^2$ considered above if the correlations of $\mathbf{X}$ up to order 6 are known.

We also note that there may be essential differences between properties of the input and output processes related by memoryless non-linear transformations, as illustrated by the following example.

Example 6. Let X be a real-valued diffusion process defined by the stochastic differential equations

$$\mathrm{d}X(t) = a(X(t))\,\mathrm{d}t + b(X(t))\,\mathrm{d}B(t), \quad t \geqslant 0, \tag{26}$$

where B is a Brownian motion. It is assumed that the functions a, b satisfy the uniform Lipschitz condition, so that the solution X of Eq. (26) exists and is unique [3, Section 4.7.1.1]. Let $Y(t) = g(X(t))$ be an output process, where $g : \mathbb{R} \to \mathbb{R}$ has a continuous second order derivative. Itô's formula [3, Section 4.6.1] applied to the mapping $X(t) \mapsto Y(t) = g(X(t))$ yields

$$\mathrm{d}Y(t) = \left(g'(X(t))a(X(t)) + \tfrac{1}{2}g''(X(t))b(X(t))^2\right)\mathrm{d}t + g'(X(t))b(X(t))\,\mathrm{d}B(t). \tag{27}$$

If g has an inverse g^{-1}, then Y is a diffusion process since $X(t)$ in the above equation can be replaced by $g^{-1}(Y(t))$. Otherwise, Y is not a diffusion process showing that the input properties can be altered in a fundamental way when mapped into an output.

3.2.2. Dynamic response

In contrast to linear systems, second-moment properties of the output $\mathbf{Y}$ of a non-linear system cannot be obtained from the first two moments of the input $\mathbf{X}$ to the system. Additional information on $\mathbf{X}$ at the level required by the classes of equivalence $\mathscr{H}_{hocm}$, $\mathscr{H}_{hoc}$, $\mathscr{H}_{sm,cd}$, or $\mathscr{H}_{sm,d}$ is still insufficient for finding the second-moment properties of $\mathbf{Y}$. Generally, the probability law of the input has to be specified completely to find even the second-moment properties of $\mathbf{Y}$. Methods of non-linear random vibration can be used to calculate properties of $\mathbf{Y}$ from properties of $\mathbf{X}$ [3, Section 7.3].

Example 7. Consider the input-output relationship

$$\mathrm{d}Y(t) = \frac{\sigma}{4}(Y(t) - \mu)^{-1/2}\,\mathrm{d}t + (Y(t) - \mu)^{3/4}\,\mathrm{d}X(t), \tag{28}$$

where $\sigma > 0$, μ are some constants and the input $X = B$ is a Brownian motion process. The input has finite moments of any order at any time $t \geqslant 0$. The stationary density,

$$f(y) = \left(\frac{\sigma}{2\pi}\right)^{1/2}(y - \mu)^{-3/2}\exp\left(\frac{\sigma}{2(y - \mu)}\right), \quad y > \mu, \tag{29}$$

of the real-valued diffusion process in Eq. (28) results form the Fokker–Planck equation for Y [3, Example 7.41, p. 493], is referred to as the Lévy density, and has no moments [7, Section 2.2.6].

This example shows that the dynamic response of some non-linear systems can differ in a fundamental way from the input to this system, as for the case of quasi-static non-linear response (Example 6).

Example 8. Let Y be defined by the stochastic differential equation

$$\mathrm{d}Y(t) = \beta Y(t)\,\mathrm{d}t + Y(t)\,\mathrm{d}X(t), \quad t \geqslant 0, \tag{30}$$

where $\beta = \alpha + \sigma^2/2$, α, σ are some constants, and the input X is known as a member of $\mathscr{C}_{in} = \mathscr{H}_{sm}$ with $E[X(t)] = 0$ and $E[X(t)X(s)] = \sigma^2(r \wedge s)$ so that $\dot{X}(t) = \mathrm{d}X(t)/\mathrm{d}t$ is a white-noise process with mean zero and covariance function $E[\dot{X}(t)\dot{X}(s)] = \sigma^2\delta(t-s)$. This formal interpretation of the white noise is common in linear random vibration. Suppose that we need to establish a criterion for the a.s. stability of the trivial stationary solution of Eq. (30). The available information on the input X is insufficient for solution. We need to consider models for X in $\mathscr{C}_{in,a}$ to be possible to find the required output properties.

Let σB and C be a scaled Brownian motion and a compound Poisson process such that $Y_1 \in L_2$, $E[Y_1] = 0$, and $\lambda E[Y_1^2] = \sigma^2$ (Eq. (9)). These processes are equal to X in the second-moment, so that they are members of $\mathscr{C}_{in}$. They are also members of $\mathscr{C}_{in,a}$ since we can calculate all properties of Y in Eq. (30) with $X = \sigma B$ and $X = C$. It can be shown that

$$Y_B(t) = Y(0)\exp\left[\left(\alpha + \sigma\frac{B(t)}{t}\right)t\right], \quad t \geqslant 0, \tag{31}$$

and

$$Y_C(t) = Y(0)\exp\left[\left(\beta + \frac{C^*(t)}{t}\right)t\right], \quad t \geqslant 0, \tag{32}$$

where $C^*(t) = \sum_{k=1}^{N(t)} \ln(1 + Y_k)$ provided that $1 + Y_1 > 0$ a.s. [3, Examples 8.55 and 8.56]. The long-term behavior of the outputs Y_B and Y_C can differ significantly. For example, $\lim_{t\to\infty} Y_B(t) = 0$ a.s. if $\alpha < 0$ since $B(t)/t$ converges a.s. to 0 as $t \to \infty$, that is, the trivial stationary solution of Eq. (30) with $X = \sigma B$ is asymptotically stable a.s. for $\alpha < 0$. For $X = C$ we have $\lim_{t\to\infty} Y_C(t) = 0$ a.s. if $\alpha + \sigma^2/2 + \lambda E[\ln(1 + Y_1)] < 0$ since $C^*(t)/t$ converges a.s. to $\lambda E[\ln(1 + Y_1)]$ as $t \to \infty$ [3, Examples 8.55 and 8.56]. Hence, $\lim_{t\to\infty} Y_C(t) = 0$ a.s. if $\alpha < \alpha^*(\lambda) = -\sigma^2/2 - \lambda E[\ln(1 + Y_1)]$.

Fig. 7 shows the dependence of $\alpha^*(\lambda)$ on λ for $\sigma = 1$ and Y_1 uniformly distributed in $(-a, a)$ with $a = \sigma\sqrt{3/\lambda}$ and $1 + Y_1 > 0$ a.s. For these properties of Y_1 the processes C and σB are equal in the second-moment sense. The plot in the figure has been obtained by Monte Carlo simulation. There is a notable difference between the long-term behavior of the solutions of Eq. (30) driven by Brownian motion and compound Poisson processes. The trivial stationary solutions of Eq. (30) with $X = \sigma B$ and $X = C$ are unstable for $\alpha > 0$ and $\alpha > \alpha^*(\lambda) > 0$, respectively. The figure suggests that $\alpha^*(\lambda)$ approaches 0 as $\lambda \to \infty$, an expected results since $C(t)$ becomes a version of σB as λ increases indefinitely [3, Section 8.7]. However, the stability regions $\alpha < 0$ and $\alpha < \alpha^*(\lambda)$ for the trivial stationary solutions Y_B and Y_C, respectively, do not coincide for $\lambda < \infty$. The stationary solutions Y_B and Y_C differ in a fundamental way for $\alpha \in (0, \alpha^*(\lambda))$. For these values of α, the

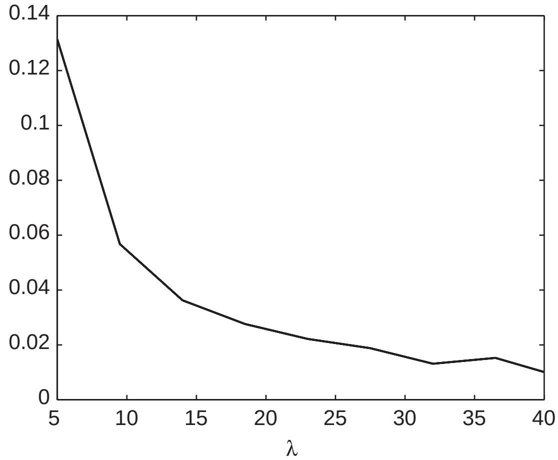

Fig. 7. Stability boundary for the trivial solution Y_C.

stationary solution Y_B is a stationary process with non-zero mean while Y_C has the property $\lim_{t\to\infty} Y_C(t) = 0$ a.s.

Example 9. Consider the response of a non-linear oscillator to a stationary input X specified partially by its marginal distribution and second-moment properties, that is, $\mathscr{C}_{in} = \mathscr{H}_{sm,d}$. The system output Y is the solution of

$$\ddot{Y}(t) + 2\zeta\nu_0\dot{Y}(t) + \xi(Y(t)) = X(t), \tag{33}$$

where $0<\zeta<1$, $\nu_0>0$, $\xi(y)=\nu_0^2 y$ for $|y|\leqslant a$, $\xi(y)=\nu_0^2[a+b(1-\mathrm{e}^{-\beta\,(|y|-a)})]\,\mathrm{sign}(y)$ for $|y|>a$, and $a, b, \nu_0>0$ are some constants.

Suppose that the marginal distribution and the correlation functions of X are given by Eqs. (14) and (15), respectively. Then the translation and diffusion processes X_T and X_D in Eqs. (11) and (12) are members of the class $\mathscr{C}_{in}$ of competing models for X. Moreover, these processes are also in $\mathscr{C}_{in,a}$ since it is possible to calculate any properties of Y from Eq. (33) with $X = X_T$ and $X = X_D$. Let Y_T and Y_D denote the output of Eq. (33) to $X = X_T$ and $X = X_D$, respectively. The processes Y_T and Y_D are not equal in the second-moment sense since Eq. (33) is a non-linear differential equation. Numerical results have been obtained for $\nu_0 = 10$, $\zeta = 0.1$, $a = 0.04$, and $b = 0.03$. Estimates of the skewness and kurtosis coefficients are $\gamma_3 = 3.03$ and $\gamma_4 = 23.24$ for Y_T, and $\gamma_3 = 2.29$ and $\gamma_4 = 12$ for Y_D, indicating a marked difference between the tails of the marginal distributions of Y_T and Y_D. This remark is consistent with Fig. 8 showing histograms of Y_T and Y_D as well as estimates of the stationary probabilities $P(Y_T(t)>y)$ and $P(Y_D(t)>y)$. The histograms and all estimates are based on 1000 output samples.

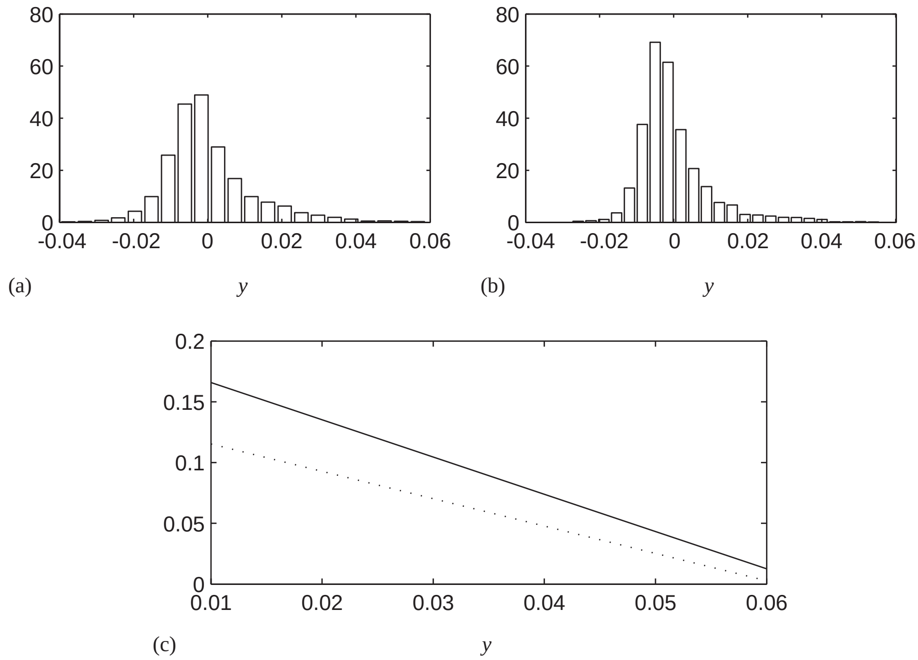

Fig. 8. Histograms of (a) Y_T and (b) Y_D and (c) estimates of the stationary probabilities $P(Y_T(t) > y)$ (solid line) and $P(Y_D(t) > y)$ (dotted line).

4. Conclusions

The available information on the random parameters in the definition of a stochastic problem is rarely sufficient for specifying the probability laws of these parameters uniquely. Generally, there exists a collection of probabilistic models consistent with the available information, referred to as the class of competing models $\mathcal{C}_{in}$.

The paper has considered perfectly known deterministic systems subjected to partially specified input processes. Generally, the information available on the input is insufficient for calculating some output properties, so that this information needs to be enhanced for solution. Since the available information can be enhanced in various ways, there may be many processes that are consistent with the available information on the input and allow the calculation of the required output properties. The collection of these processes was denoted by $\mathcal{C}_{in,a}$. It has been shown that some output properties depend strongly on the particular member in $\mathcal{C}_{in,a}$ used to describe the input. This dependence can have significant practical implications if, for example, the input is represented by an arbitrarily selected member of $\mathcal{C}_{in,a}$, rather than using decision or other methods for selecting a member of $\mathcal{C}_{in,a}$ that describes the input in an optimal sense.

Examples involving simple linear and non-linear systems have been used to (1) illustrate the flow of information from input to output and (2) quantify the dependence of some output properties on the input models in $\mathcal{C}_{in,a}$. It has been found that for linear systems, some partial information on the input maps into similar information on the output. This is not the case with non-linear systems. The calculation of most output properties for these systems requires a detailed characterization of the input. Results in the paper suggest to consider a relatively broad class of competing models for input and select an optimal model for the input from this class based on rational methods rather than heuristic considerations.

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